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LETTER TO THE EDITOR

Braid modules

Florin Constantinescu† and Mirko Lüdde‡§

† Institut für Angewandte Mathematik, Universität Frankfurt, Robert Mayer Strasse 10, D-6000 Frankfurt/M 1, Federal Republic of Germany

‡ Physikalisches Institut, Universität Bonn, Nussallee 12, D-5300 Bonn, Federal Republic of Germany

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Abstract. We introduce representations of the braid group B_{n-1} in the ring $\mathrm{Mat}(Z[B_n], n-1)$ of matrices whose elements belong to the ring $Z[B_n]$ of the braid group B_n. Iteration of this representation provides the tool for finding the braiding and monodromy matrices associated to multiple line integrals of holomorphic functions. We consider applications in two-dimensional conformal field theory.

In this letter we are going to compute the action of the braid group onto multiple line integrals of a broad class of holomorphic functions [1]. Compared with related work [2–5] on this problem we consider it as a simplification because we are able to develop the framework in a simpler algebraic setting. Of course due to its origin all the presented algebra has a geometrical interpretation which, however, will be presented elsewhere [6].

In the following for any $n \in N$, B_n will denote the permuting braid group generated by the set $\{\tau_i, 1 \leqslant i \leqslant n-1\}$, with relations

$$\tau_i \tau_j = \tau_j \tau_i \quad | i - j | \geqslant 2 \qquad \tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1} \,. \tag{1}$$

For later convenience we abbreviate

$$\tau_{i,j} = \begin{cases} \tau_i \tau_{i+1} \cdots \tau_{j-1} & i < j \\ \tau_{i-1} \tau_{i-2} \cdots \tau_j & j < i \\ 1 & j = i \end{cases} \tag{2}$$

$$\mu_{i,j} = \tau_{i,j} \tau_{j,i} \qquad \vartheta_{i,j} = \tau_{j-1,i}^{-1} \qquad \mu_{j-1,j} \tau_{j-1,i} \,.$$

Let us now consider two sets $\{x_i, 1 \leqslant i \leqslant n-1\}$ and $\{w_j, 1 \leqslant j \leqslant n-2\}$ and out of them construct in a formal way modules $X_0^{(1)}$ and $W_0^{(1)}$, respectively. $X_0^{(1)}$ as an (additive) Abelian group is the group of formal finite linear combinations of the form

$$\sum_j \alpha_j x_j \beta_j \tag{3}$$

§ E-mail address: luedde@pib1.physik.uni-bonn.de

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where the left coefficients α_j belong to the ring $Z[B_n]$ but the right coefficients are taken from $Z[B_{n-1}]$. (We recall that the ring $Z[B_n]$ is defined by the formal sums $\sum_{\alpha\in B_n} n_\alpha \cdot \alpha$, with only finitely many non-vanishing integers $n_\alpha \in \mathbf{Z}$ and the product between such elements is the induced one coming from B_n extended by Z-linearity.) The group $X_0^{(1)}$ of formal linear combinations thus defined is given the structure of a left module over $Z[B_n]$ and at the same time that of a right module over $Z[B_{n-1}]$ by the left and right action

$$Z[B_n] \times X_0^{(1)} \ni \left(\gamma, \sum_j \alpha_j x_j \beta_j\right) \mapsto \sum_j \gamma\alpha_j x_j \beta_j \tag{4}$$

$$X_0^{(1)} \times Z[B_{n-1}] \ni \left(\sum_j \alpha_j x_j \beta_j, \delta\right) \mapsto \sum_j \alpha_j x_j \beta_j \delta \tag{5}$$

respectively. The construction of the module $W_0^{(1)}$ proceeds in a similar way by considering the linear combinations $\sum_{k=1}^{n-2} \alpha_k w_k \beta_k$ with free generators w_k instead of x_j.

The relations

$$x_j \tau_i = \sum_k B(i)_j^k x_k \tag{6}$$

$$w_l \tau_i = \sum_m R(i)_l^m w_m \tag{7}$$

with matrices $B(i) \in \mathrm{Mat}(Z[B_n], (n-1))$ and $R(i) \in \mathrm{Mat}(Z[B_n], (n-2))$ given by

$$(B(i)_j^k) = 1_{(i-1,i-1)} \cdot \tau_{i+1} \oplus \begin{pmatrix} \alpha_i & \beta_i \\ \tau_{i+1} & 0 \end{pmatrix} \oplus 1_{(n-2-i,n-2-i)} \cdot \tau_{i+1} \tag{8}$$

$$(R(i)_l^m) = 1_{(i-2,i-2)} \cdot \tau_{i+1} \oplus \begin{pmatrix} \tau_{i+1} & \tau_{i+2,i} & 0 \\ 0 & -\tau_i \tau_{i+2,i} & 0 \\ 0 & \tau_{i,i+2} & \tau_i \end{pmatrix} \oplus 1_{(n-i-3,n-i-3)} \cdot \tau_i \tag{9}$$

with the restrictions

$$1 \leqslant i \leqslant n-2 \qquad 1 \leqslant j,k \leqslant n-1 \qquad 1 \leqslant l,m \leqslant n-2 \tag{10}$$

and abbreviations

$$\alpha_i = \tau_{i+1}(1 - \vartheta_{1,i+1}\vartheta_{1,i+2}\vartheta_{1,i+1}^{-1}) \qquad \beta_i = \tau_{i+1}\vartheta_{1,i+1} \tag{11}$$

can now be consistently imposed on the modules. We call the resulting quotients $X^{(1)}$ and $W^{(1)}$, respectively. Consistency follows from the fact that, as one can easily check by computation, the matrices $B(i)$ and $R(i)$ constitute matrix representations of Artin's braid group B_{n-1}. This means

$$\sum_k B(i)_l^k B(j)_k^m = \sum_k B(j)_l^k B(i)_k^m \qquad \text{for} \quad |i-j| \geqslant 2 \tag{12}$$

$$\sum_{k,l} B(i)_j^k B(i+1)_k^l B(i)_l^m = \sum_{k,l} B(i+1)_j^k B(i)_k^l B(i+1)_l^m \tag{13}$$

besides the existence of the inverse matrices

$$(B^{-1}(i)_j^k) = \tau_{i+1}^{-1}\left(1_{(i-1)\times 2} \oplus \begin{pmatrix} 0 & 1 \\ \alpha_i' & \beta_i' \end{pmatrix} \oplus 1_{(n-2-i)\times 2}\right) \tag{14}$$

with $\alpha_i' = \vartheta_{1,i+2}^{-1}$, $\beta_i' = \vartheta_{1,i+2}^{-1}(\vartheta_{1,i+1} - 1)$, whereas the inverse of $R(i)$ is simply obtained by the braid group automorphism $\tau_l \mapsto \tau_l^{-1}$.

One should notice that the matrices $B(i)$ are (braid-ring-valued) generalizations of the Burau representation of the braid group, whereas the matrices $R(i)$ correspond to the reduced Burau representation [7]. We claim that these matrices yield a faithful representation of the braid group, since they are intimately connected with Artin's theorem on faithful braid representations in automorphism groups of free groups [7]. This, however, will be discussed elsewhere [6].

We now come to the crucial point of our considerations. In a similar way to the construction of $X_0^{(1)}$ (and $W_0^{(1)}$) it is possible to construct further modules $X_0^{(2)}, \ldots, X_0^{(n-1)}$ (similarly for $W_0^{(l)}$) by taking (for fixed index l, $1 \leqslant l \leqslant n-1$) the set of formal additively linear products

$$\alpha_1 x_{i_1} \alpha_2 x_{i_2} \ldots \alpha_l x_{i_l} \alpha_{l+1} \tag{15}$$

with $\alpha_k \in Z[B_{n+1-k}]$ and $1 \leqslant i_k \leqslant n-k$ and then turning this set into the freely generated left module over $Z[B_n]$ and right module over $Z[B_{n-l}]$. Again we impose the relations (7) and (6) onto the free modules $X_0^{(l)}, W_0^{(l)}$, which now can be used iteratively. This is exemplified by writing

$$x_{i_1} x_{i_2} \ldots x_{i_l} \tau_k = \sum_{m_1} x_{i_1} \ldots x_{i_{l-1}} B(k)_{i_l}^{m_1} x_{m_1} \tag{16}$$

$$= \sum_{m_1, m_2} x_{i_1} \ldots x_{i_{l-2}} B^{(2)}(k)_{i_l, i_{l-1}}^{m_1, m_2} x_{m_2} x_{m_1} \tag{17}$$

$$= \sum_{m_1, \ldots, m_l} B^{(l)}(k)_{i_l, \ldots, i_1}^{m_1, \ldots, m_l} x_{m_l} \ldots x_{m_1} \,. \tag{18}$$

The matrices $B^{(l)}(i)$ carrying multiple indices are obtained from those introduced before simply by replacing the generators τ_i, which occur in the matrix elements by the corresponding representation matrices $B(i)$ and iterating this procedure until enough indices are obtained. It should be clear that for $2 \leqslant l \leqslant n-1$ the matrices $B^{(l)}(i)$ again constitute representations of the braid groups B_{n-l} (B_1 is the trivial group with one element), just because the replacements $\tau_i \mapsto B(i)$ respect the properties of the generators τ_i. We set $X = \oplus_{l=1}^{n-1} X^{(l)}$ and $W = \oplus_{l=1}^{n-1} W^{(l)}$ and call the so-obtained graded $Z[B_n]$ left modules (in which the submodules of grade l are also right $Z[B_{n-l}]$ modules) the unreduced and reduced Artin modules, respectively.

Without proof (which can be filled in by the reader, again by mere computation) we note that further relations can be imposed on X and W. For example the relations

$$w_i w_j = w_{j+1} w_i \qquad \text{for} \quad i \leqslant j \tag{19}$$

are consistent in the reduced Artin module W. Consistency here means that the submodule in $W^{(2)}$ spanned by the elements $w_i w_j - w_{j+1} w_i$ for $i \leqslant j$ is invariant under the right action of $Z[B_{n-2}]$ and thus can be set to zero. In X it is possible to impose

$$\sum_{i=1}^{n-1} \vartheta_{1,1} \ldots \vartheta_{1,i} x_i = 0 \tag{20}$$

such that there are as many independent generators for $X^{(1)}$ as there are for $W^{(1)}$. Vice versa one can add a generator w_0 to $W^{(1)}$ and obtain an imbedding $X \subset W$ by setting

$$x_i = \sum_{j=0}^{i-1} \tau_{j+1,1}^{-1}(1 - \vartheta_{j+1,i}) w_j \, . \tag{21}$$

We wrote down the former relations just because they hold in the realization of the Artin modules which we are now going to introduce. Let

$$M_n^{>} = \{(z_1, \ldots, z_n) \qquad \text{if} \quad i < j \quad \text{then} \quad |\, z_i \,| > |\, z_j \,| \quad -\mathrm{i}\pi < \arg(z_k) < \mathrm{i}\pi\} \tag{22}$$

be the simply connected subset of C^n obtained by ordering the variables z_k with respect to their absolute value and cutting the complex planes in which they take their values along the negative real axis. Let $\{f_j, j \in J\}$ be a family of complex functions each member of which is holomorphic in $M_n^{>}$. Thus singularities can only occur if two variables approach each other $z_i \to z_j$. These functions can be continued analytically to the universal covering of the space $M_n = \{(z_1, \ldots, z_n),$ if $i \neq j$ then $z_i \neq z_j\}$. If we choose a point $P \in M_n^{>}$ (e.g. $P = (1, 1/2, \ldots, 1/n)$) and denote by γ some path in M_n starting at $\gamma(0) = P$ and ending at $\gamma(1) = z \in M_n$, then the expression $f_j(z, \gamma)$ denotes the uniquely defined analytic continuation of the complex function f_j from (a neighborhood of) the point $P \in M_n^{>}$ to the point $z \in M_n$ along the path γ. The result only depends on the homotopy class of the path due to the monodromy theorem.

We now introduce a representation of the braid group B_n in terms of linear operators $\hat{\tau}_i$ acting on the functions f_j: for every $z \in M_n^{>}$ we set

$$(\hat{\tau}_i f_j)(z) = f_j(z, \gamma_i(z)) \tag{23}$$

for a path $\gamma_i(z)$ in M_n running from P to z by first interchanging the neighbouring components P_i and P_{i+1} in mathematically positive orientation and then connecting the resulting point with the point $(z_1, \ldots, z_{i-1}, z_{i+1}, z_i, z_{i+2}, \ldots, z_n) = t_i(z)$ on a path lying completely in the set $t_i M_n^{>}$. The symbol t_i denotes the transposition which exchanges the ith and $(i+1)$th components of a tupel. In this way the path $\gamma_i(z)$ is determined up to homotopy equivalence. Products of generators τ_i act according to the equation

$$(\widehat{\tau_i \tau_j)(f_k})(z) = (\hat{\tau}_j(\hat{\tau}_i(f_k)))(z) = f_k(z, \gamma_i(P) \circ t_i(\gamma_j(z))) \tag{24}$$

where $\gamma_i(P) \circ t_i(\gamma_j(z))$ denotes the path obtained by first running through $\gamma_i(P)$ and then through $t_i(\gamma_j(z))$. The transposition ensures that the endpoint of $\gamma_i(P)$

(which is $t_i(P)$) is equal to the starting point of $t_i(\gamma_j(z))$. Notice that the domain of all the generated functions is kept fixed to be $M_n^>$. This domain therefore is the image of a sheet of the Riemann surface of the functions f_j, where the chart is determined by the generators τ_i. Thus the braid generators τ_i, τ_j^{-1} climb up and down on the Riemann surface, respectively.

Next we have to introduce linear operators corresponding to the generators x_i and w_j of $X^{(1)}$ and $W^{(1)}$. For this purpose for every $z = (z_1, \ldots, z_{n-1}) \in M_{n-1}^>$ we let $\gamma(x_j)(z)$ denote a path $t \mapsto \gamma(x_j)(z)(t) \in C - \{z_1, \ldots, z_{n-1}\}$ which is constructed as follows. If $z = (1, \ldots, 1/(n-1))$ it starts at $\mathrm{Re}(z_j) - i\infty$, runs parallel to the imaginary axis until it comes near z_j, circumvents z_j on a small (not containing points z_i, $i \neq j$) circle oriented positively and then runs back to infinity, again parallel to the negative imaginary axis. For general $z \in M_{n-1}^>$ the path is obtained by deforming the previous one by some continuous deformation of $(1, \ldots, 1/(n-1))$ into z which stays in $M_{n-1}^>$. On the other hand $\gamma(w_j)(z)$ is a path starting at z_j and running to z_{j+1} such that up to the endpoints of the path the tupel $(z_1, \ldots, z_j, \gamma(w_j)(z)(t), z_{j+1}, \ldots, z_{n-1})$ is lying in $M_n^>$.

Now for every $z \in M_{n-1}^>$ we set

$$(\hat{x}_i f_j)(z) = \int_{\gamma(x_i)(z)} f_j(t, z_1, \ldots, z_{n=1})\, \mathrm{d}t \tag{25}$$

$$(\hat{w}_i f_j)(z) = \int_{\gamma(w_i)(z)} f_j(z_1, \ldots, z_j, t, z_{j+1}, \ldots, z_{n-1})\, \mathrm{d}t\,. \tag{26}$$

We assume that the behaviour of the functions at infinity and at its singularities is sufficiently mild such that the integrals exist and result in functions being holomorphic in $M_{n-1}^>$.

We now claim (postponing the proof to a later publication [6]) that among the operators $\hat{\tau}_i, \hat{x}_j, \hat{w}_k$ the relations (7) and (6) hold and (for suitable functions) the additional relations (19)–(21). In particular, by use of these relations it is possible to compute the representations of the braid group carried by the families $\{\hat{x}_{i_l} \ldots \hat{x}_{i_1} f_j,\ 1 \leqslant i_k \leqslant n-k, j \in J\}$ of integrated functions if there is a representation on the unintegrated family $\{f_j, j \in J\}$. In the sense of [1] the relations encode the combined action of the braid group onto the homology of M_n and onto the functions f_j.

These points have been worked out in more detail in [6, 8], so here we will proceed by giving two simple examples.

Example 1. Let

$$f_a(z_1, \ldots, z_n) = \prod_{i<j} (z_i - z_j)^{a_{i,j}} \tag{27}$$

with $a = (a_{1,2}, a_{1,3}, \ldots, a_{n-1,n})$ and let $N(i,j) = (i-1)\cdot(n - i/2) + j - i$ be the position of $a_{i,j}$ in the tupel a. To every generator τ_i associate the canonical transposition $t_i = (i \mapsto i+1, i+1 \mapsto i)$ and the permutation π_i of $N(n-1,n)$-tupels acting as

$$\pi_i a = (a_{t_i 1, t_i 2}, \ldots, a_{i,i+1}, \ldots, a_{t_i(n-1), t_i n})\,. \tag{28}$$

Then the C-linear hull of the f_b, where b is created from a by the described permutations, is invariant under the braid group

$$(\hat{\tau}_i f_b)(z_1, \ldots, z_n) = \mathrm{e}^{\mathrm{i}\pi b_{i,i+1}} \cdot f_{\pi_i b}(z_1, \ldots, z_n) \tag{29}$$

i.e. B_n is linearly represented on this vector space by the matrices $(\mathrm{e}^{\mathrm{i}\pi a_{j,j+1}} \cdot \delta_{b,\pi_j a})_{a,b}$.

If we now choose the $a_{i,j}$ so as to obtain a hypergeometric type function by integration $(a_{i,j} \mapsto \delta_{i,1} \cdot a_{j-1})$,

$$\int_{\gamma(x_i)(z)} \prod_{k=1}^{n} (t - z_k)^{a_k} \, \mathrm{d}t = (\hat{x}_i f_a)(z_1, \ldots, z_n) \tag{30}$$

with $a = (a_1, \ldots, a_n)$, and $1 \leqslant i \leqslant n$ (note the change of notation of a) we obtain a braid group representation of B_n on the vector space spanned by $\{\hat{x}_i f_{pa}, 1 \leqslant i \leqslant n, p \in P_n = \text{symmetric group on } n \text{ elements}\}$ in terms of very simple matrices

$$\hat{x}_i f_a \mapsto \hat{\tau}_j \hat{x}_i f_a = \sum_{1 \leqslant k \leqslant n, b \in P_n a} M(j)_{i,a}^{k,b} \hat{x}_k f_b \tag{31}$$

$$M(j)_{i,a}^{k,b} = \hat{B}(j,a)_i^k \delta_{t_j a}^b \tag{32}$$

where $\hat{B}(j,a)$ results from $B(j)$ by the replacements

$$\tau_{j+1} \mapsto 1 \qquad \vartheta_{1,k} \mapsto \exp(2\pi\mathrm{i} \cdot (t_j a)_{k-1}) \,. \tag{33}$$

The second map constitutes a one-dimensional representation of the monodromy group of f_b regarded as function in one variable. In this way we also obtain the Gassner representation of the coloured braid groupoid, since due to the second Artin relation of the uncoloured braid group the matrices obey the coloured braid relation

$$\hat{B}(j,a)\hat{B}(j+1,t_j a)\hat{B}(j,t_{j+1}t_j a) = \hat{B}(j+1,a)\hat{B}(j,t_{j+1}a)\hat{B}(j+1,t_j t_{j+1}a) \,. \tag{34}$$

For $a_i = a_j$ (all i,j) the Gassner representation reduces to the Burau representation of the braid group.

Choosing $a_{i,j} = 2 \cdot \varrho_i \cdot \varrho_j$ the functions f_a in (27) become the expectation value of a product of free vertex operators $V_\varrho(z) =: \exp(\mathrm{i}\varrho\phi(z))$ in the massless Gaussian measure. By integrating 'screening'-vertex-operators over contours $y_i = \sum_{j=i}^{n-2} \tau_{i+1,j+1}(1 - \mu_{j+1,n})w_j$ (this transformation corresponds to an endomorphism of the module $W^{(1)}$) one can calculate the braiding properties of chiral intertwining vertex operators of minimal conformal models, as done in [9] by means of a different strategy.

Example 2. Multiple two-dimensional integrals over monodromy-invariant bilinear forms of holomorphic and antiholomorphic functions can be converted into multiple line integrals, which under certain conditions factorize into holomorphic and antiholomorphic integrals [10].

The general formula [11, 12] can be written in compact form with the help of the braid module:

$$Q\hat{P}^m\,(F\bar{F}) = \sum_{i,j}\int_{D^m} Q^{i,j}\,F_i(z)\bar{F}_j(\bar{z})\mathrm{d}z_n \wedge \mathrm{d}\bar{z}_n \wedge \ldots \wedge \mathrm{d}\bar{z}_{n-m+1} \tag{35}$$

with $P = P_1 + P_2$. P_1 contains unfactorized integrals over the boundary of D, whereas P_2 consists of a sum of products of (anti-)holomorphic factors

$$P_2 = \sum_{i,j=0}^{n-2} c^{i,j}\cdot w_i \cdot \bar{w}_j = \sum_{i=0}^{n-2} d^i \cdot y_i \cdot \bar{y}_i \text{ (modulo } Q) \tag{36}$$

with bilinear forms

$$c^{i,j} = \begin{cases} (\tau_{i+1,n}^{-1} - \tau_{n,i+1})\cdot\bar{\tau}_{n,j+1} & i \geqslant j \\ \tau_{i+1,n}^{-1}\cdot(\bar{\tau}_{n,j+1} - \bar{\tau}_{j+1,n}^{-1}) & i < j \end{cases}$$

$$d^l = \begin{cases} \tau_{l+1,n}^{-1}\cdot(\tau_l^{-1}\cdot(1-\mu_{l,n})^{-1}\cdot\tau_l - (1-\mu_{l+1,n})^{-1})\cdot\bar{\tau}_{l+1,n}^{-1} & l \neq 0 \\ \tau_{1,n}^{-1}\cdot(\mu_{1,n}-1)^{-1}\cdot\bar{\tau}_{1,n}^{-1} & l = 0\,. \end{cases}$$

Here Q denotes the bilinear form combining the (anti-) holomorphic (in $M_n^>$) functions $F_i, \overline{F}_j$ to functions on the plane without monodromy. D is assumed to be a compact region of the complex plane, containing the singularities of the F_i (which reside among the z_j) except for ∞. Therefore the integral is an improper one, which is assumed to exist by eventually performing an analytic continuation in the exponents of the singularities z_j. Otherwise one had to remove neighbourhoods of the z_j from D and the boundary of D would become larger. For properly chosen functions the limit $D \to C$ can be performed such that there is no boundary left at all and the contribution coming from P_1 vanishes. An example is $\int_{\mathbb{C}} |z|^a|z-1|^b|z-t|^c\,\mathrm{d}z \wedge \mathrm{d}\bar{z}$, where $a, b, c > -2$ and $a+b+c < -2$.

The generators y_i introduced in example 1 can now be recognized as the ones diagonalizing the matrix $c^{i,j}$ (at the prize of introducing formal inverses $(1-\mu_{l,n})^{-1}$). In fact all the different kinds of contours appearing in the literature on conformal field theory [10, 9, 13] are related by module endomorphisms and can be chosen for convenience.

With help of the relations of the graded module it is possible to calculate the normal form of P_2^m where all analytic continuations are carried out before the integrations [6]. This is the generalization of the methods of [14, 11] to compute the perturbation series around conformal field theory models. By choosing either the contours x_i or y_j it is possible to extract either the short- or long-distance singular contributions of the two-dimensional integrals of the perturbation series into the braiding factors.

To conclude this letter let us remark that the presented braid module makes it possible to iteratively produce new braid representations from given ones and in particular it solves the problem of finding the braid (and monodromy) representations carried by line integrals of holomorphic functions.

Furthermore the presented framework yields the tools for dealing with the holomorphic factorization formula for multiple two-dimensional integrals given in the second example.

Beyond these applications there exists the possibility of generating new solutions to the Yang–Baxter equation [8] as well as defining new algebras generalizing the familiar q-deformations $U_q(g)$ of semisimple Lie algebras g [15].

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References

[1] Mostow G D 1987 *Bull. Am. Math. Soc.* **16** 2
[2] Aomoto K 1987 *J. Math. Soc. Japan* **39** No 2 191
[3] Felder G and Wieczerkowski C *ETH-Zürich Preprint* ETH TH90-52, NSF-ITP 90-203
[4] Kohno T 1988 *Adv. Stud. Pure Math.* **16**
[5] Lawrence R J 1990 *Commun. Math. Phys.* **135** 141
[6] Constantinescu F and Lüdde M in preparation
[7] Birman J 1974 *Braids Links and Mapping Class Groups (Ann. Math. Stud. 82)* (Princeton, NJ: Princeton University Press)
[8] Lüdde M and Toppan F 1992 *Phys. Lett.* **288B** 321
[9] Felder G, Fröhlich J and Keller G 1989 *Commun. Math. Phys.* **124** 647
[10] Dotsenko Vl S 1988 *Adv. Stud. Pure Math.* **16**
Dotsenko Vl S and Fateev V A 1985 *Nucl. Phys.* B **251** 691
[11] Chaselon P, Constantinescu F and Flume R 1991 *Phys. Lett.* **257B** 63
[12] Mathur S D 1990 *Harvard Preprint* HUTMP-90-B/299
[13] Gomez C and Sierra G 1991 *Nucl. Phys.* B **352** 791
[14] Dotsenko Vl S 1989 *Nucl. Phys.* B **314** 687
[15] Lüdde M 1992 *Preprint* BONN-HE-92-9 (*J. Math. Phys.* at press)